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Technical Notes

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Optimal Geometry of Self-Field Magnetoplasmadynamic Thrusters

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Introduction

IN a recent article Toki[1] applies the Optimal Control Theory (OCT) to define the optimal geometry of a quasi-one-dimensional self-field magnetoplasmadynamic (MPD) arcjet. The thruster model is highly idealized; in particular, the energy equation is not included in the relevant equations. A similar model had been used by Kuriki et al.,[2] who applied it to different assigned geometries of the thruster channel. Their article pointed out that the magnetic flux density should be zero at the channel exit, but this end condition cannot be attained in a constant-area thruster when the voltage applied between the electrodes is low, as the derivative of the magnetic flux vanishes. The authors have carried out the same computations as in Toki's article, but were not able to fulfill the end condition concerning the magnetic flux density. This difficulty is not explicitly declared by Toki, but is implicitly confirmed by the figures in the article; in this case, it is a result of the extremely high value of the magnetic flux derivative that makes the channel end section singular.

The OCT application to a simple model of a MPD thruster is more complex than it seems in Toki's article. This Note applies OCT more rigorously but obtains less encouraging results. Toki has suggested a channel geometry to maximize the exit velocity for specified operating conditions; his results also show that the current distribution is moderately uniform over the entire channnel length and does not exhibit the peaks that usually occur at both the inlet and exit sections. The optimal geometry presents a convergent portion, followed by a highly divergent duct. The inapplicability of the quasi-one-dimensional assumption and practical reasons would suggest imposing a limit to the channel cross section, i.e., a constraint on the control variable. The necessary conditions for maximum final velocity in the presence of such a constraint are found in this Note and are used to obtain the thruster geometry; under this assumption a current peak appears in the exit region. A similar peak at the channel end section, with infinite current density, is exhibited by the rigorous solution of the unconstrained optimization problem, which is also described in the following sections.

Problem Formulation

The formulation of the problem in the present article strictly follows Toki's assumptions.[1] A quasi-one-dimensional flow in a constant-width channel is considered; variables are function of the abscissa x along the channel length. No external magnetic field is present; the electrical conductivity has a constant value; the plasma pressure is ignored; and, therefore, the energy conservation equation is neglected. The problem is ruled by two differential equations for the plasma velocity u and the magnetic flux B:

$$\frac{\mathrm{d}u}{\mathrm{d}x} = R_m(V - AuB)B \tag{1}$$

$$\frac{\mathrm{d}B}{\mathrm{d}x} = -R_m\left(\frac{V}{A} - uB\right) \tag{2}$$

The equations have been expressed in nondimensional form by means of the same normalization as in Toki's article.[1] The channel height A is the problem control variable. Two parameters R_m and V are present in Eqs. (1) and (2); the former is the magnetic Reynolds number, which is representative of the square of the total current; the latter, when R_m has been specified, represents the discharge voltage. The absolute value of Eq. (2) right-hand side is the nondimensional current density j.

At the channel inlet (subscript i) $B_i = 1$ is a necessary consequence of the normalization, whereas at the exit (subscript e) $B_e = 0$ is required by the physics of the problem. The plasma velocity at the channel inlet is usually specified: the value assumed in Toki's computations $u_i = 0.5$ is generally adopted throughout. Both the parameters R_m and V can be assigned in advance when the channel geometry is sought ($R_m = 10$ and $V = 4.7$ in the following). Either V or R_m must be left free if the channel profile is assigned.

The channel geometry, which maximizes the final velocity, is searched for by using an indirect method, that is, by applying OCT. The authors prefer and usually adopt the Mayer formulation[3] instead of the Lagrange formulation adopted by Toki. The performance index $\varphi = u_e$ is maximized. The Hamiltonian is defined as

$$H = R_m[\lambda_u(V - uBA)B - \lambda_B(V/A - uB)] \tag{3}$$

The optimal height or cross-section area of the constant-width channel is obtained by nullifying the Hamiltonian partial derivative with respect to A, thus obtaining

$$A^2 = \frac{\lambda_B V}{\lambda_u u B^2} \tag{4}$$

The derivatives of the adjoint variables λ_u and λ_B are provided by the Euler–Lagrange equations

$$\frac{\mathrm{d}\lambda_u}{\mathrm{d}x} = R_m(\lambda_u BA - \lambda_B)B \tag{5}$$

$$\frac{\mathrm{d}\lambda_B}{\mathrm{d}x} = R_m[\lambda_u(2uBA - V) - \lambda_B u] \tag{6}$$

The nondimensional channel length is unit. To solve the differential system of Eqs. (1), (2), (5), and (6), the fourth boundary condition $\lambda_{ue} = 1$ is provided by OCT.[4] The problem is homogeneous, as far as the adjoint variables are concerned, and the last

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boundary condition could be replaced by a more convenient one, that, for instance, would prescribe the initial value of either adjoint variable.

Results

The problem that has been outlined in the preceding section is well posed, but its numerical solution cannot be achieved. The authors have not been able to fulfill the end condition $B_e = 0$, but they have obtained the same exit area $A_e = 34.70$, final velocity $u_e = 2.8758$, and smooth current profile as in Toki's article by enforcing $B_e = 0.0309$. The corresponding geometry, which is here conveniently termed as "Toki's geometry," presents an initially convergent profile that becomes highly divergent to reach a very large end section. On the contrary, the inlet area is limited ($A_i = 8.16$) because of the relatively large value of the initial velocity.

Practical reasons suggest imposing a limit to the channel cross section, i.e., a constraint on the control variable. The position of the optimal control problem is unchanged. In the most complex case when the maximum value of the channel cross section is small (at least with reference to the specified value of the inlet velocity) the channel length is divided into three parts. The channel height is an unconstrained control variable in the middle portion, whereas the constraint $A = A_{\max}$ is enforced in the external regions. Control, state, and adjoint variables are continuous at the junctions. The same initial and final conditions, as stated in the preceding section, are still valid. The continuity of the control variable is sufficient to provide the lengths of the constant-area portions. An additional constraint has implicitly been assumed: the channel profile cannot again be convergent after the throat.

The optimal solution of this new problem does not present any numerical difficulty. By enforcing $A_{\max} = 10$, the final velocity is slightly reduced ($u_e = 2.8495$), and a current-density peak appears at the channel exit (Fig. 1). When $A_{\max}$ is reduced, this current peak increases; nevertheless, it has not been introduced by the area constraint but is inherent to the problem, as it is related to the necessary condition $B_e = 0$. This conclusion has been confirmed by a long time-consuming process that has overcome the numerical difficulties to eventually achieve $B_e = 0.0043$, using the unconstrained-area model. A similar peak in the current density appears (Fig. 2); one should also note the impressive profile of the channel.

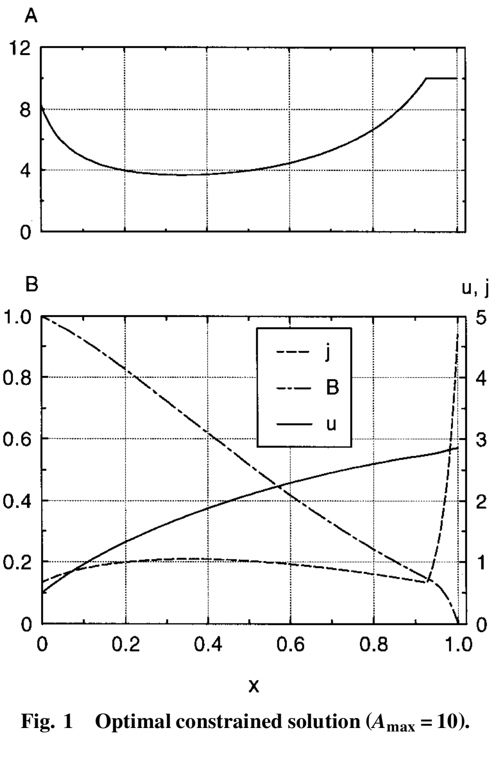

Fig. 1 Optimal constrained solution ($A_{\max} = 10$).

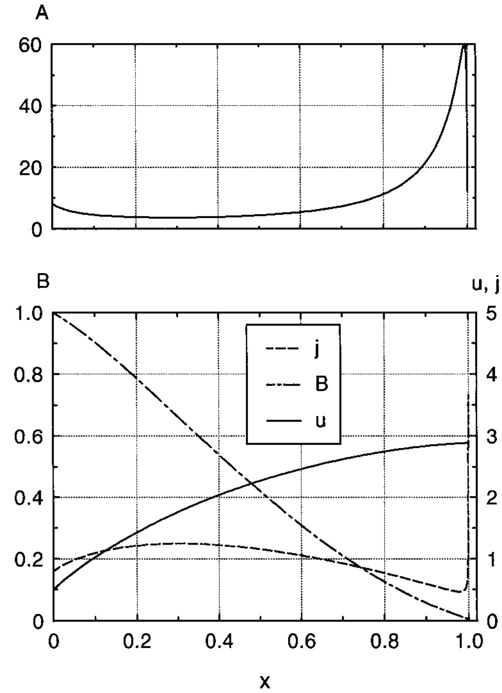

Fig. 2 Unconstrained solution ($B_e = 0.0043$).

This result clears the features of the unconstrained problem solution that is optimal according to the adopted mathematical model. The adjoint variable λ_B becomes nil before B reaches zero, and, according to Eq. (4), the optimal cross section is also zero. The current density, that is, the B derivative, is not finite, and a discontinuity of the magnetic flux density occurs in the same section to fulfill the physical end condition $B_e = 0$. From the numerical point of view, the optimal solution is obtained by using $\lambda_{Be} = 0$ as the end condition, instead of $B_e = 0$. The new boundary condition pertains to the optimal channel with a free final value of the magnetic flux density. The optimal geometry can be obtained by integrating Eqs. (1), (2), (5), and (6) backwards from the exit to the inlet section. The correctness of this solution is supported by the highest computed value of the final velocity ($u_e = 2.8870$). Further evidence is provided by the results of the constrained problem, as Pontryagin's Maximum Principle would suggest $A < A_{\max}$ just before the exit section, where λ_B is even negative.

Discussion

The application of OCT to this quasi-one-dimensional model of an MPD thruster produces results that, in the absence of an area constraint, do not agree with the initial hypothesis. In particular, the channel cross section varies too rapidly, even in Toki's geometry. The correct analytical solution clearly shows the shortcomings that are related to neglecting the energy equation. The last point should make one cautious of accepting the solution of the constrained-area problem, which, instead, appears to be reasonable in many respects.

The results do not lead to much optimism as far as the current distribution is concerned. One easily obtains

$$j_e = R_m V / A_e \tag{7}$$

$$j_i = R_m (V / A_i - u_i) \tag{8}$$

and the current density is reduced by enlarging the inlet and/or exit cross sections and by increasing the plasma initial velocity. The inability to enforce $B_e = 0$, when the cross-section area is not bounded, causes an illusory reduction of the current density at the channel exit.[1] If Toki's geometry is assigned in advance to an MPD thruster and an analysis of performance is carried out for $R_m = 10$, $B_e = 0$ is easily attained together with $V = 4.7$. Numerical computations

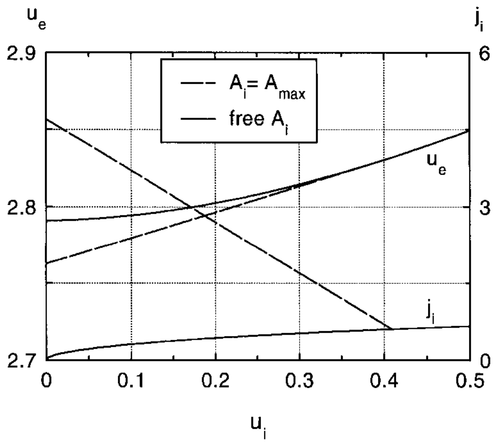

Fig. 3 Influence of the inlet velocity ($A_{max} = 10$).

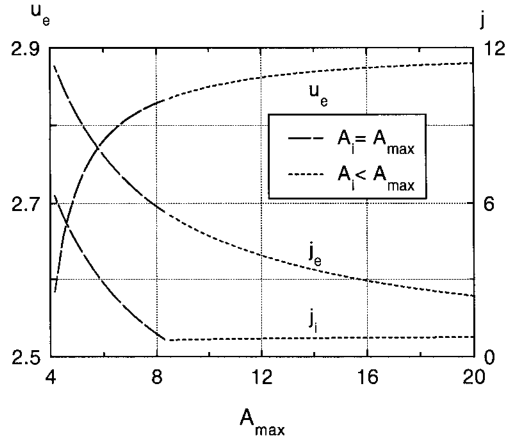

Fig. 4 Influence of the area limit ($u_i = 0.5$).

and Eq. (7) concordantly provide $j_e = 1.355$, whereas a small benefit on the exit velocity ($u_e = 2.8835$) is obtained. The inaccuracy of Toki's results is confirmed by searching for the optimal geometry of a thruster that presents a constant-area appendage with the same $A_{max} = 34.70$, as in Toki's geometry. The numerical solution provides $u_e = 2.8858$ and, obviously, $j_e = 1.355$.

The constraint $A_e = A_{max}$ is in fact always necessary; to be consistent, one should at least impose an equal limit on the initial cross section. The same value $A_{max} = 10$ is considered in Fig. 3, which points out the influence of the inlet velocity; the channel begins with a constant-area duct if $u_i < 0.4$. The initial peak of the current density is more plausibly removed by specifying a high u_i than by exploiting the optimization process. Unfortunately, the increase of the initial velocity does not produce the same increase in the exit velocity, and the engine thrust decreases; the engine efficiency, which depends on the square of the plasma velocity, is instead improved.

The same initial velocity $u_i = 0.5$ is considered in Fig. 4; a larger value of A_{max} produces better performance and reduces both the current-density peaks. The left extremes of the curves in Fig. 4 ($A_{max} = 4.17$) correspond to a constant-area channel that becomes convergent-divergent as soon as A_{max} is increased. The improvement of the thruster characteristics gradually diminishes, and $A_{max} = 10$ seems to be a suitable compromise between engine performance and dimensions.

Conclusions

The theory of optimal control has been applied to the analysis of a MPD thruster by using a simple quasi-one-dimensional model that neglects the energy equation. A correct solution of the unconstrained problem has been presented and discussed. The introduction of a constraint concerning the maximum value of the channel cross-section area has provided more realistic results; nevertheless, a more precise model that takes the energy equation into account would be opportune. The optimization of the plasma final velocity does not imply the smoother distribution of the current density that has been asserted in recent literature. The practical goal of OCT application is only the fast design of the channel geometry that ensures high specific impulse for specified values of the discharge voltage and total current. In the authors' opinion the improvement of theoretical knowledge is, however, the main outcome of an indirect optimization procedure.

References

[1]Toki, K., "Optimal Control of Quasi-One-Dimensional Self-Field Magnetoplasmadynamic Arcjet Flowfields," *Journal of Propulsion and Power*, Vol. 13, No. 1, 1997, pp. 157–161.

[2]Kuriki, K., Kunii, Y., and Shimizu, Y., "Idealized Model for Plasma Acceleration in an MHD Channel," *AIAA Journal*, Vol. 21, No. 3, 1983, pp. 322–326.

[3]Cicala, P., *An Engineering Approach to the Calculus of Variations*, 2nd ed., Levrotto and Bella, Turin, Italy, 1997, pp. 10–12.

[4]Bryson, A. E., and Ho, Y.-C., *Applied Optimal Control* (rev. printing), Hemisphere, Washington, DC, 1975, pp. 47–49.

Transonic Compressor Influences on Upstream Surface Pressures with Axial Spacing

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Introduction

GAS turbines are a vital energy source for both industrial and military applications, with recent research focusing on identifying high cycle fatigue unsteady flow mechanisms. There is a constant need for improved understanding of the flow physics through the various components. Greater understanding of these mechanisms provides manufacturers with the ability to achieve higher levels of performance and more efficient systems. As the level of technology increases, there are continuous demands on gas turbine engines to achieve greater durability, reduced noise levels, size, and, of course, greater thrust. A considerable portion of recent research involves the unsteady interaction between adjacent blade rows in both compressor and turbine sections.

The objective of this research is to investigate and quantify the fundamental vane/blade interaction phenomena of the upstream traveling potential forcing function from a downstream rotor in a compression system. This is accomplished by performing a series of

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